**Open point of the study about the existence of pairs of complementary spaces based on dashes**

**Hypothesis:**

1. n 1 and n 2 prime with n 1 < n 2 ;
2. n integer greater than 1

**Thesis:**

1. there exist two multiples of n 1 : n 1 \*h and n 1 \*k (with h and k > 0) such that n 1 \*h -1 and n 1 \*k +1 are not divisible by n 2
2. h+k must be equal to a fixed n greater than 1

Proof of the Thesis a)

There exist infinitely many h and infinitely many k such that n 1 \*h -1 and n 1 \*k +1 are not divisible by n 2 .

In ascending order of h and k, n 1 \*h -1 and n 1 \*k +1 are not divisible by n 2 , being smaller than it, for all h < and for all k < .

We now prove that for infinitely many values of h and k greater than and , n 1 \*h -1 and n 1 \*k +1 are not divisible by n 2 and we can write:

1. n 1 \*h -1 = h'\* n 2 +r h with 0 <r h <n 2
2. n 1 \*k +1 = k'\* n 2 +rk with 0 <rk < n 2

In fact from 1) we see that for every h Ꜫ N greater than and such that [h\* n 1 ] mod n2 is different from 1, setting r h = [h\* n 1 ] mod n2 – 1 (that is, setting [h\* n 1 ] and [r h +1] congruent withn 2 ), there will always exist an h' Ꜫ N which satisfies 1).

Similarly from 2) we see that for every k Ꜫ N such that [k\* n 1 ] mod n2 is different from -1, setting r k = [k\* n 1 ] mod n2 + 1 (that is [k\* n 1 ] and [rk -1] congruous withn 2 ), there will always exist a k' Ꜫ N satisfying 2).

The two conditions: [h\* n 1 ] mod n2 different from 1 and [k\* n 1 ] mod n2 different from -1 always hold for the h and k smaller than respectively h0 and k0, and for the greater ones the conditions exclude the possibility that r h or rk can be equal to 0.

Proof of Thesis b)

We now have to prove that among the infinitely many h and k that satisfy Thesis a) there are also some that satisfy the condition h + k =n.

We have seen that, in order to satisfy Thesis A), [h\* n 1 ] mod n2 must be different from 1 and [k\* n 1 ] mod n2 must be different from -1, i.e. the two conditions must be met:

1. [h\* n 1 ] mod n2 ≠ 1
2. [k\* n 1 ] mod n2 ≠ -1 ≠ n 2 - 1

If we now impose the condition that h + k = n, two consequences follow:

1. that both h and k (both greater than 0, with n greater than 1 by assumption) belong to the interval [1, n-1]
2. that h and k are complementary with respect to n so that in modular arithmetic the following relation exists between the modules n 2 of their products by n 1 :

 [k\*n 1 ] mod n2 = [(nh)\*n 1 ] mod n2 = [n\*n 1 -h\*n 1 ] mod n2 = [n\*n 1 ] mod n2 --

[h\*n 1 ] mod n2

with the consequence that also [h\*n 1 ] mod n2 and [k\*n 1 ] mod n2 are complementary with respect to [n\*n 1 ] mod n2, so if n is a multiple of n 2 we will have that [n\*n 1 ] mod n2 = 0 = n 2 and the occurrence or otherwise of condition 3) will also imply that of condition 4); if, on the other hand, n2 does not divide n, it is necessary to verify that both conditions are met.

**In case A), therefore that n2 divides n,** to satisfy condition 3), and as mentioned therefore also condition 4), it is enough to prove that there is at least one hi such that:

[ \* n 1 ] mod n2 ≠ 1 with Ꜫ {1,2, ……. n-1}

now for each h i and each h i+1 (with 1 < i <n-1) it is possible to write:

[ \* n 1 ] mod n2 = [hi \* n1 ± n1]mod n2 = [hi\* n1] mod n2 ± [n1]mod n2.

and this means that if [ \* n 1 ] mod n2 = 1 it will not be [ \* n 1 ] mod n2 (or vice versa) being [n 1 ] mod n2 = n 1 and n 1 ≠ n 2 .

In conclusion and or and will satisfy conditions 3) and 4).

**In case B) instead, where n2 does not divide n,** we have to check that both conditions 3) and 4) are verified. Let us then consider any two values of h and k complementary with respect to n and the moduli n2 of their products by n 1 :

[h i \*n 1 ] mod n2 and [k j \*n 1 ] mod n2 with i, j Ꜫ {1,2, ……. n-1} and j = n – i

Now for each hi , hi+1 andk j-1 (with 1 < i <n-1) it is possible to consider the following relations:

 [h i \*n 1 ] mod n2 = [h i+1 \*n 1 ] mod n2 – [n 1 ] mod n2

[ \* n 1 ] mod n2 + [n 1 ] mod n2 \* n 1 ] mod n2

If [hi \* n 1 ] mod n2 = 1 we have that:

[h i+1 \*n 1 ] mod n2 = [h i \*n 1 ] mod n2 + [n 1 ] mod n2 ≠ 1

being [n 1 ] mod n2 ≠ 0

on the other hand we will have:

[k j-1 \*n 1 ] mod n2 = [k j \*n 1 ] mod n2 – [n 1 ] mod n2 = [n\*n 1 ] mod n2 - [h i \*n 1 ] mod n2 – [n 1 ] mod n2 = [n 1 ] mod n2 \* [n-1] mod n2 – 1

and so **if we exclude that n 2 divides (n-1)** will result[k j-1 \*n 1 ] mod n2 ≠ -1 with the conclusion that h i+1 and k j-1 will satisfy conditions 3) and 4).

If instead [k j \*n 1 ] mod n2 = -1 we will have that:

[k j+1 \*n 1 ] mod n2 = [k j \*n 1 ] mod n2 + [n 1 ] mod n2 ≠ -1

being [n 1 ] mod n2 ≠ 0

on the other hand we will have:

[h i-1 \*n 1 ] mod n2 = [h i \*n 1 ] mod n2 – [n 1 ] mod n2 = [n\*n 1 ] mod n2 - [k j \*n 1 ] mod n2 – [n 1 ] mod n2 = [n 1 ] mod n2 \* [n-1] mod n2 + 1

and so **if we exclude that n 2 divides (n-1)** will result[h i-1 \*n 1 ] mod n2 ≠ 1 with the conclusion that h i-1 and k j+1 will satisfy conditions 3) and 4).

If instead [h i \*n 1 ] mod n2 ≠ 1 and [k i \*n 1 ] mod n2 ≠ 1 they will be indeed hi andki to satisfy conditions 3) and 4).

**Finally, in case C) that n2 does not divide n but divides n – 1** we always consider any two values of h and k complementary with respect to n and the moduli n 2 of their products by n 1 :

[h i \*n 1 ] mod n2 and [k j \*n 1 ] mod n2 with i, j Ꜫ {1,2, ……. n-1} and j = n – i

Now for each hi , hi+2 and kj-2 (with 2 < i <n-2) it is possible to consider the following relations.

If [hi \* n 1 ] mod n2 = 1 we have that:

[h i+2 \*n 1 ] mod n2 = [h i \*n 1 ] mod n2 + 2\*[n 1 ] mod n2 ≠ 1

where [n 1 ] mod n2 ≠ 0 and 2\* n 1 ≠ n 2

on the other hand we will have:

[k j-1 \*n 1 ] mod n2 = [k j \*n 1 ] mod n2 – [n 1 ] mod n2 = [n\*n 1 ] mod n2 - [h i \*n 1 ] mod n2 – [n 1 ] mod n2 = [n 1 ] mod n2 \* [n-1] mod n2 – 1

[k j-2 \*n 1 ] mod n2 = [n 1 ] mod n2 \* [n-1] mod n2 – 1 - [n 1 ] mod n2 = [n 1 ] mod n2 \* [n-2] mod n2 – 1 ≠ -1

being [n-2] mod n2 ≠ 0 since n-2 is not divisible by n 2 since n-1 is already divisible by it, and being [n 1 ] mod n2 ≠ 0

In conclusion, h i+2 and k j-2 will satisfy conditions 3) and 4).

If instead [k j \*n 1 ] mod n2 = -1 we will have that:

[k j+2 \*n 1 ] mod n2 = [k j \*n 1 ] mod n2 + 2\*[n 1 ] mod n2 ≠ -1

where [n 1 ] mod n2 ≠ 0 and 2\* n 1 ≠ n 2

on the other hand we will have:

[h i-1 \*n 1 ] mod n2 = [h i \*n 1 ] mod n2 – [n 1 ] mod n2 = [n\*n 1 ] mod n2 - [k j \*n 1 ] mod n2 – [n 1 ] mod n2 = [n 1 ] mod n2 \* [n-1] mod n2 + 1

[h i-2 \*n 1 ] mod n2 = [n 1 ] mod n2 \* [n-1] mod n2 + 1 - [n 1 ] mod n2 = [n 1 ] mod n2 \* [n-2] mod n2 + 1 ≠ 1

being [n-2] mod n2 ≠ 0 since n-2 is not divisible by n2 since n-1 is already divisible by it, and being [n 1 ] mod n2 ≠ 0

In conclusion, hi-2 and kj+2 will satisfy conditions 3) and 4).

If instead [h i \*n 1 ] mod n2 ≠ 1 and [k i \*n 1 ] mod n2 ≠ -1 they will be indeed hi andki to satisfy conditions 3) and 4).

For n ≥ 5 the thesis of the theorem is verified for each of the three considered cases of n: n 2 divides n, n 2 divides neither n nor (n-1), n 2 does not divide n but divides (n-1).

Instead for n=2 the theorem is verified since both h and k are equal to 1 and therefore less than h 0 and k 0 respectively; for n=3 the theorem is verified according to the case A) or to that B) according to the divisibility of n by n 2 ; finally, for n=4 the theorem is verified on the basis of case B).